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Existence results for impulsive neutral stochastic functional integro-differential inclusions with nonlocal initial conditions[☆]

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ABSTRACT

In this paper, we prove the existence of mild solutions for a class of impulsive neutral stochastic functional integro-differential inclusions with nonlocal initial conditions and resolvent operators. Sufficient conditions for the existence are derived with the help of the fixed point theorem for multi-valued operators due to Dhage and the fractional power of operators. An example is provided to illustrate the theory.

1. Introduction

In this paper, we shall consider the existence of mild solutions for the following impulsive neutral stochastic functional integro-differential inclusions with nonlocal conditions:

$$\begin{cases} \mathrm{d}\left[x(t) - g\left(t, x\left(h_1(t)\right), \int_0^t a\left(t, s, x(h_2(s))\right)\mathrm{d}s\right)\right] \in A\left[x(t) + \int_0^t f(t-s)x(s)\mathrm{d}s\right]\mathrm{d}t \\ \quad + F(t, x(h_3(t)))\mathrm{d}w(t), \quad t \in J := [0, b],\, t \neq t_k, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k\left(x(t_k^-)\right), \quad k = 1, 2, \ldots, m, \\ x(0) + h(x) = x_0, \end{cases} \tag{1}$$

where A is the infinitesimal generator of a compact, analytic resolvent operator $S(t)$, $t \geq 0$ in the Hilbert space H. Suppose $\{w(t) : t \geq 0\}$ is a given K-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q \geq 0$ and $L(K; H)$ denotes the space of all bounded linear operators from K into H. Let $h_i : J \to J$, $i = 1, 2, 3$ and $f(t)$, $t \in J$, be a bounded linear operator. Here, $I_k \in C(H, H)$ $(k = 1, 2, \ldots, m)$ are bounded functions. Furthermore, the fixed times t_k satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_m < b$, $x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of $x(t)$ at $t = t_k$. $\Delta x(t_k) = x\left(t_k^+\right) - x\left(t_k^-\right)$ represents the jump in the state x at time t_k, where I_k determines the size of the jump. The random variable $x_0 \in H_\alpha$ satisfies $E\|x_0\|_\alpha^2 < \infty$, and g, a, F, h are given functions specified later.

The theory of impulsive integro-differential equations or inclusions has become an active area of investigation due to their applications in the fields such as mechanics, electrical engineering, medicine biology, ecology and so on. One can see

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[1–3] and references therein. Several authors have established the existence results of mild solutions for these equations (see [4–7] and references therein).

The starting point of this paper is the following delay semilinear differential inclusion which introduced in [8]:

$$\begin{cases} x' - Ax \in F(t, x(\tau(t))), & t \in [0, b], \\ x(0) + h(x) = x_0. \end{cases}$$

Furthermore, Ezzinbi et al. [9] considered the following neutral partial differential equations in α-norm:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}[x(t) - F(t, x(h_1(t)))] = -A[x(t) - F(t, x(h_1(t)))] + G(t, x(h_2(t))), & t \in [0, b], \\ x(0) + h(x) = x_0. \end{cases}$$

Recently, Chang et al. [10] extended the results of [9] to the following impulsive neutral integro-differential inclusions with nonlocal initial conditions in α-norm and proved the existence of the solutions by a fixed point theorem for condensing multi-valued maps:

$$\begin{cases} \mathrm{d}[x(t) - F(t, x(h_1(t)))] \in A\left[x(t) + \displaystyle\int_0^t f(t-s)x(s)\mathrm{d}s\right]\mathrm{d}t + G(t, x(h_2(t))), & t \in J := [0, b],\ t \neq t_k, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k\left(x(t_k^-)\right), & k = 1, 2, \ldots, m, \\ x(0) + h(x) = x_0. \end{cases}$$

The nonlocal Cauchy problem was considered by Byszewski [11] and the importance of nonlocal conditions in different fields has been discussed in [12,10,13] and the references therein. For example, in [12] the author described the diffusion phenomenon of a small amount of gas in a transparent tube by using the formula

$$h(x) = \sum_{i=1}^{p} c_i x(t_i),$$

where $c_i, i = 0, 1, \ldots, p$ are given constants and $0 < t_0 < t_1 < \cdots < t_p < b$. In this case the above equations allows the additional measurement at $t_i, i = 0, 1, \ldots, p$. In the past several years theorems about existence, uniqueness and stability of differential and functional differential abstract evolution Cauchy problem with nonlocal conditions have been studied by many authors, one can see [10] and references therein.

In addition, the nonlinear integro-differential equations with resolvent operators serve as an abstract formulation of partial integro-differential equations that arise in many physical phenomena. One can see [14] and references therein.

The deterministic models often fluctuate due to noise, which is random or at least appears to be so. Therefore, we must move from deterministic problems to stochastic problems. As the generalization of classic impulsive integro-differential equations or inclusions, impulsive neutral stochastic functional integro-differential equations or inclusions have attracted the researchers great interest. And some works have done on the existence results of mild solutions for these equations (see [15,16] and references therein).

To the best of our knowledge, there is no work reported on the existence of mild solutions for the impulsive neutral stochastic functional integro-differential inclusions with nonlocal initial conditions and resolvent operators, and the aim of this paper is to close the gap. In this paper, motivated by the previously mentioned papers, we will study this interesting problem. Sufficient conditions for the existence are given by means of the fixed point theorem for multi-valued mapping due to Dhage [17] and the fractional power of operators. Especially, the known results appeared in [10,18] are generalized to the stochastic settings. An example is provided to illustrate the theory.

2. Preliminaries

For more details on this section, we refer the reader to Da Prato and Zabczyk [19]. Throughout the paper $(H, \|\cdot\|_H)$ and $(K, \|\cdot\|_K)$ denote two real separable Hilbert spaces. In case without confusion, we just use $\langle\cdot,\cdot\rangle$ for the inner product and $\|\cdot\|$ for the norm.

Let $(\Omega, \mathcal{F}, P; \mathbf{F})$ $\left(\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}\right)$ be a complete filtered probability space satisfying that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. An H-valued random variable is an $\mathcal{F}$-measurable function $x(t) : \Omega \to H$ and the collection of random variables $S = \{x(t, \omega) : \Omega \to H | t \in J\}$ is called a stochastic process. Generally, we just write $x(t)$ instead of $x(t, \omega)$ and $x(t) : J \to H$ in the space of S. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of K. Suppose that $\{w(t) : t \geq 0\}$ is a cylindrical K-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$, denote $\mathrm{Tr}(Q) = \sum_{i=1}^{\infty} \lambda_i = \lambda < \infty$, which satisfies that $Qe_i = \lambda_i e_i$. So, actually, $w(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} w_i(t) e_i$, where $\{w_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Wiener processes. We assume that $\mathcal{F}_t = \sigma\{w(s) : 0 \leq s \leq t\}$ is the σ-algebra generated by w and $\mathcal{F}_T = \mathcal{F}$. Let $\psi \in L(K, H)$ and define

$$\|\psi\|_Q^2 = \mathrm{Tr}(\psi Q \psi^*) = \sum_{n=1}^{\infty} \left\|\sqrt{\lambda_n}\psi e_n\right\|^2.$$

If $\|\psi\|_Q < \infty$, then ψ is called a Q-Hilbert–Schmidt operator. Let $L_Q(K, H)$ denote the space of all Q-Hilbert–Schmidt operators $\psi : K \to H$. The completion $L_Q(K, H)$ of $L(K, H)$ with respect to the topology induced by the norm $\|\cdot\|_Q$ where $\|\psi\|_Q^2 = \langle\psi, \psi\rangle$ is a Hilbert space with the above norm topology.

Let $A : D(A) \to H$ be the infinitesimal generator of a compact, analytic resolvent operator $S(t)$, $t \geq 0$. Let $0 \in \rho(A)$. Then, it is possible to define the fractional power A^α for $0 < \alpha \leq 1$, as a closed linear operator with its domain $D(A^\alpha)$ being dense in H. We denote by H_α the Banach space $D(A^\alpha)$ endowed with the norm

$$\|x\|_\alpha = \|(-A)^\alpha x\|.$$

Furthermore, we have the following properties appeared in [7].

Lemma 1. *The following two properties hold:*

(i) *If $0 < \beta < \alpha \leq 1$, then $H_\alpha \hookrightarrow H_\beta$ and the embedding is continuous and compact whenever the resolvent operator of A is compact;*

(ii) *For every $0 < \alpha \leq 1$, there exists C_α such that*

$$\|(-A)^\alpha S(t)\| \leq \frac{C_a}{t^\alpha}, \quad t > 0. \tag{2}$$

Let $L_2(\Omega, \mathcal{F}_{t,H})$denote the Hilbert space of all $\mathcal{F}_t$-measurable square integrable random variables with values in H. Let $L_2^{\mathcal{F}}([0, b], H)$ be the Hilbert space of all square integrable and $\mathcal{F}_t$-measurable processes with values in H.

$$\mathcal{B}([0, b]) = \big\{x : [0, b] \to H_\alpha, x_k \in C(J_k, H_\alpha) \text{ and there exist } x(t_k^-) \quad \text{and} \quad x(t_k^+) \text{ with}$$
$$x(t_k) = x(t_k^-), k = 0, 1, 2, \ldots, m, x(0) + h(x) = x_0\big\}.$$

Let $\mathcal{B}$ denote the Banach space $\mathcal{B}([0, b], L_2(\Omega, \mathcal{F}, H))$, the family of all $\mathcal{F}_t$-measurable, $\mathcal{B}([0, b])$-valued random variables x with the norm

$$\|x\|_{\mathcal{B}} = \max\{\|x_k\|_{J_k},\ k = 0, 1, 2, \ldots, m\},$$

where x_k is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m$, with

$$\|x_k\|_{J_k} = \sup_{s\in J_k} \left(E\|x_k(s)\|_\alpha^2\right)^{1/2}.$$

Let $L_2^0(\Omega, \mathcal{B})$ denote the family of all $\mathcal{F}_0$-measurable, $\mathcal{B}$-valued random variables $x(0)$.

We use the notations $\mathcal{P}_{cl}(H)$ for the family of all subsets of H and denote

$$\mathcal{P}_{cl}(H) = \{Y \in \mathcal{P}(H) : Y \text{ is closed}\}, \qquad \mathcal{P}_{bd}(H) = \{Y \in \mathcal{P}(H) : Y \text{ is bounded}\},$$
$$\mathcal{P}_{cv}(H) = \{Y \in \mathcal{P}(H) : Y \text{ is convex}\}, \qquad \mathcal{P}_{cp}(H) = \{Y \in \mathcal{P}(H) : Y \text{ is compact}\}.$$

In what follows, we briefly introduce some facts on multi-valued analysis. For details, one can see [20].

A multi-valued map $\Gamma : H \to \mathcal{P}(H)$ is convex (closed) valued, if $\Gamma(x)$ is convex (closed) for all $x \in H$. Γ is bounded on bounded sets if $\Gamma(B) = \cup_{x\in B} \Gamma(x)$ is bounded in H, for any bounded set B of H, that is, $\sup_{x\in B} \sup\{\|y\| \in \Gamma(x)\} < \infty$.

Γ is called upper semicontinuous (u.s.c. for short) on H, if for any $x \in H$, the set $\Gamma(x)$ is a nonempty, closed subset of H, and if for each open set B of H containing $\Gamma(x)$, there exists an open neighborhood N of x such that $\Gamma(N) \subseteq B$.

Γ is said to be completely continuous if $\Gamma(B)$ is relatively compact, for every bounded subset $B \subseteq H$.

If the multi-valued map Γ is completely continuous with nonempty compact values, then Γ is u.s.c. if and only if Γ has a closed graph, i.e., $x_n \to x$, $y_n \to y$, $y_n \in \Gamma(x_n)$ imply $y \in \Gamma(x)$.

Γ has a fixed point if there is $x \in H$ such that $x \in \Gamma(x)$.

A multi-valued map $\Gamma : J \to \mathcal{P}_{cl}$ is said to be measurable if for each $x \in H$, the mean-square distance between x and $\Gamma(t)$ is measurable.

Definition 2. The multi-valued map $F : J \times H \to \mathcal{P}_{bd,cl,cv}(H)$ is said to be L^2-Carathéodory if

(i) $t \mapsto F(t, v)$ is measurable for each $v \in H$;

(ii) $v \mapsto F(t, v)$ is u.s.c. for almost all $t \in J$;

(iii) for each $q > 0$, there exists $h_1 \in L^1(J, \mathbf{R}_+)$ such that

$$\|F(t, v)\|^2 := \sup_{f\in F(t,v)} E\|f\|^2 \leq h_q(t), \quad \text{for all } \|v\|_{\mathcal{B}}^2 \leq q \text{ and for a.e. } t \in J.$$

Then, we have the following lemma due to Lasota and Opial [21].

Lemma 3. *Let I be a compact interval and H be a Hilbert space. Let F be an L^2-Carathéodory multi-valued map with $N_{F,x} \neq \emptyset$ and let Γ be a linear continuous mapping from $L^2(I, H)$ to $C(I, H)$. Then, the operator*

$$\Gamma \circ N_F : C(I, H) \to \mathcal{P}_{cp,cv}(H), \qquad x \mapsto (\Gamma \circ N_F)(x) = \Gamma(N_{F,x}),$$

is a closed graph operator in $C(I, H) \times C(I, H)$, where $N_{F,x}$ is known as the selectors set from F, is given by

$$\sigma \in N_{F,x} = \left\{\sigma \in L^2\left(L(K, H)\right) : \sigma(t) \in F(t, x) \text{ for a.e. } t \in J\right\}.$$

Now, we give the fixed point theorem due to Dhage [17], which is our main tool.

Theorem 4. *Let $B(0, r)$ and $B[0, r]$ denote respectively the open and closed balls in a Hilbert space H centered at the origin and of radius r and let $\Phi_1 : H \to \mathcal{P}_{bd,cl,cv}(H)$ and $\Phi_2 : B[0, r] \to \mathcal{P}_{cp,cv}(H)$, two multi-valued operators satisfying*
(i) *Φ_1 is a contraction, and*
(i) *Φ_2 is u.s.c. and completely continuous.*

Then, either
(1) *the operator inclusion $x \in \Phi_1 x + \Phi_2 x$ has a solution, or*
(2) *there exists an $x \in H$ with $\|x\| = r$ such that $\lambda x \in \Phi_1 x + \Phi_2 x$ for some $\lambda > 1$.*

Now, we give knowledge on the resolvent operator which appeared in [14].

Lemma 5. *A family of bounded linear operators $S(t) \in \mathcal{P}(H)$, $t \in J$ is called a resolvent operator for*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A\left[x(t) + \int_0^t f(t-s)x(s)\,\mathrm{d}s\right],$$

if
(i) *$S(0) = I$, the identity operator on H;*
(ii) *for all $u \in H$, $S(t)u$ is continuous for $t \in J$;*
(iii) *$S(t) \in \mathcal{P}(Y)$, $t \in J$, where Y is the Banach space formed from $D(S)$ endowed with the graph norm. For $y \in Y$, $S(\cdot)y \in C^1(J, H) \cap C(J, Y)$ and*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}S(t)y &= A\left[S(t)y + \int_0^t f(t-s)S(s)y\,\mathrm{d}s\right]\\ &= S(t)Ay + \int_0^t S(t-s)AF(s)y\,\mathrm{d}s, \quad t \in J.\end{aligned}$$

3. Main result

Before stating and proving the main result, we present the definition of the mild solution to the system (1).

Definition 6. A stochastic process $x \in \mathcal{B}$ is called a mild solution of (1) if
- $x_0, h \in L_2^0(\Omega, \mathcal{B})$;
- $x(0) + h(x) = x_0$;
- $\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k\left(x(t_k^-)\right)$, $k = 1, 2, \ldots, m$;
- $x(t) \in H$ has càdlàg paths on $t \in J$ a.s., and there exists a function $\sigma \in N_{F,x(h_3(t))}$ such that

$$\begin{aligned}x(t) &= S(t)\left[x_0 - h(x) - g(0, x(h_1(0)), 0)\right] + g\left(t, x(h_1(t)), \int_0^t a(t, s, x(h_2(s)))\,\mathrm{d}s\right)\\ &\quad + \int_0^t AS(t-s)g\left(s, x(h_1(s)), \int_0^s a(s, \tau, x(h_2(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s\\ &\quad + \int_0^t AS(t-s)\int_0^s f(s-\tau)g\left(\tau, x(h_1(\tau)), \int_0^\tau a(\tau, \delta, x(h_2(\delta)))\,\mathrm{d}\delta\right)\mathrm{d}\tau\,\mathrm{d}s\\ &\quad + \int_0^t S(t-s)\sigma(s)\,\mathrm{d}w(s) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)), \quad t \in J.\end{aligned} \tag{3}$$

Now, for some $\alpha \in (0, 1)$, we assume the following assumptions:

(H1) A is the infinitesimal generator of a compact, analytic resolvent operator $S(t)$, $t \geq 0$ in the Hilbert space H and there exist constants M_1, M_2 and M_3 such that

$$\|S(t)\|^2 \leq M_1, \quad t \in J, \qquad \|A^{-\beta}\|^2 \leq M_2, \qquad \|f(t)\|^2 \leq M_3.$$

(H2) $a : D \times H_\alpha \to H_\alpha$, $D = \{(t, s) \in J \times J : t \geq s\}$ is a continuous function and there exists a constant $M_4 > 0$ such that for all $t \in J$, $x, y \in H_\alpha$

$$\left\|\int_0^t [a(t, s, x) - a(t, s, y)]\,\mathrm{d}s\right\|_\alpha^2 \leq M_4\|x - y\|_\alpha^2.$$

(H3) There exist constants $0 < \beta < 1$ and M_5 such that $g : J \times H_\alpha \times H_\alpha \to H_\beta$ is a continuous function, and $A^\beta g : J \times H_\alpha \times H_\alpha \to H_\alpha$ satisfies the following Lipschitz condition, that is, for any $s, t \in J$, $x_1, y_1, x_2, y_2 \in H_\alpha$ such that

$$\left\|A^\beta g(s, x_1, y_1) - A^\beta g(t, x_2, y_2)\right\|_\alpha^2 \leq M_5\left[|s - t| + \|x - y\|_\alpha^2 + \|y_1 - y_2\|_\alpha^2\right].$$

(H4) $I_k \in C(H_\alpha, H_\alpha)$ and there exist continuous nondecreasing functions $J_k \in \mathbf{R} \to \mathbf{R}$ such that for every $x \in H_\alpha$

$$\|I_k(x)\|_\alpha^2 \le J_k(\|x\|_\alpha^2).$$

(H5) $h_i \in C(J, J)$, $i = 1, 2, 3$.

(H6) The multi-valued map $F : J \times H_\alpha \to \mathcal{P}_{bd,cl,cv}(L(K, H))$ is an L^2-Carathéodory function satisfies the following conditions:

(i) For each $t \in J$, the function $F(t, \cdot) : H_\alpha \to \mathcal{P}_{bd,cl,cv}(L(K, H))$ is u.s.c.; and for each $x \in H_\alpha$, the function $F(\cdot, x)$ is measurable. And for each fixed $x \in \mathcal{B}$, the set

$$F_{N,x} = \left\{\sigma \in L^2(L(K, H)) : \sigma(t) \in F(t, x) \text{ for a.e. } t \in J\right\}$$

is nonempty;

(ii) For each positive number $l > 0$, there exists a positive function $\mu(l)$ independent on l such that

$$\sup_{E\|x\|_\alpha^2 \le l} \|F(t, x)\|^2 \le \mu(l),$$

where $\|F(t, x)\|$ is defined as Definition 2.

(H7) $h : \mathcal{B} \to H_\alpha$ is completely continuous and there exist positive constants M_6 and M_7 such that

$$\|h(x)\|_\alpha^2 \le M_6\|x\|_{\mathcal{B}}^2 + M_7.$$

(H8) There exists a real number $r > 0$ such that

$$r > \frac{L_1 + \frac{6b^{1-2\alpha}}{1-2\alpha}\mathrm{Tr}(Q)\mu(r) + 6m^2M_1\sum_{k=1}^{m} J_k(r)}{1 - L_2},$$

where

$$L_1 = 18(E\|x_0\|_\alpha^2 + M_6M_7 + 2M_2c_2) + 12M_2(2c_1M_5 + c_2) + 12b(1 + bM_3)C_{1-\beta}^2\frac{b^{2\beta}}{2\beta - 1}(2c_1M_5 + c_2),$$

$$L_2 = 18(M_6 + 2M_2M_5) + 12M_2M_5(1 + 2M_4) + 12b(1 + bM_3)M_5C_{1-\beta}^2\frac{b^{2\beta}}{2\beta - 1}(1 + 2M_4),$$

$$c_1 = b\sup_{(t,s)\in D}\|a(t, s, 0)\|_\alpha^2,\ c_2 = \sup_{t\in J}\|A^\beta g(t, 0, 0)\|_\alpha^2.$$

The main result of this paper is the following theorem.

Theorem 7. *Let $x(0) \in L_2^0(\Omega, \mathcal{B})$. If the assumptions* (H1)–(H8) *hold and*

$$L_0 = 4M_5\left[M_2(1 + M_1 + M_4) + b(1 + M_4)(1 + bM_3)\frac{b^{2\beta-1}}{2\beta - 1}\right] < 1, \tag{4}$$

then the system (1) admits at least one mild solution on J.

Proof. Consider the operator $\Phi : \mathcal{B} \to \mathcal{P}(\mathcal{B})$ defined by

$$\begin{aligned}\Phi(x) = \Bigg\{ u \in \mathcal{B} : u(t) &= g\left(t, x(h_1(t)), \int_0^t a(t, s, x(h_2(s)))\,\mathrm{d}s\right) + S(t)\left[x_0 - h(x) - g(0, x(h_1(0)), 0)\right] \\ &+ \int_0^t AS(t - s)g\left(s, x(h_1(s)), \int_0^s a(s, \tau, x(h_2(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s \\ &+ \int_0^t AS(t - s)\int_0^s f(s - \tau)g\left(\tau, x(h_1(\tau)), \int_0^\tau a(\tau, \delta, x(h_2(\delta)))\,\mathrm{d}\delta\right)\mathrm{d}\tau\,\mathrm{d}s \\ &+ \int_0^t S(t - s)\sigma(s)\,\mathrm{d}w(s) + \sum_{0<t_k<t} S(t - t_k)I_k(x(t_k^-)),\ t \in J, f \in N_{F,x(h_3(t))}\Bigg\}.\end{aligned} \tag{5}$$

It is clear that the fixed points of Φ are mild solutions of the system (1). Let

$$\begin{aligned}\Phi_1(x) = \Bigg\{ u \in \mathcal{B} : u(t) &= g\left(t, x(h_1(t)), \int_0^t a(t, s, x(h_2(s)))\,\mathrm{d}s\right) - S(t)g(0, x(h_1(0), 0)) \\ &+ \int_0^t AS(t - s)g\left(s, x(h_1(s)), \int_0^s a(s, \tau, x(h_2(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s \\ &+ \int_0^t AS(t - s)\int_0^s f(s - \tau)g\left(\tau, x(h_1(\tau)), \int_0^\tau a(\tau, \delta, x(h_2(\delta)))\,\mathrm{d}\delta\right)\mathrm{d}\tau\,\mathrm{d}s\Bigg\},\end{aligned}$$

and

$$\Phi_2(x) = \left\{ u \in \mathcal{B} : u(t) = S(t)\left[x_0 - h(x)\right] + \int_0^t S(t-s)\sigma(s)\,\mathrm{d}w(s) + \sum_{0<t_k<t} S(t-t_k)I_k\left(x\left(t_k^-\right)\right),\ t \in J \right\}.$$

In what follows, we prove that the operators Φ_1 and Φ_2 satisfy all the conditions of Theorem 4. In the sequel, let $B_l = \{x \in \mathcal{B}, E\|x\|_\alpha^2 \le l\}$. We give the proof in the following seven steps:

Step 1. Φ_1 is a contraction.

Let $x_1, x_2 \in B_l$, then by the assumptions, we have

$$\begin{aligned}
E\|\Phi_1(x_1)(t) - \Phi_1(x_2)(t)\|_\alpha^2 &\le 4E\left\|S(t)g(0, x_1(h_1(0)), 0) - g(0, x_2(h_1(0)), 0)\right\|_\alpha^2 \\
&+ 4E\left\| g\left(t, x_1(h_1(t)), \int_0^t a(t, s, x_1(h_2(s)))\right)\mathrm{d}s - g\left(t, x_2(h_1(t)), \int_0^t a(t, s, x_2(h_2(s)))\right)\mathrm{d}s \right\|_\alpha^2 \\
&+ 4E\left\| \int_0^t AS(t-s)\left(g\left(s, x_1(h_1(s)), \int_0^s a(s, \tau, x_1(h_2(\tau)))\,\mathrm{d}\tau\right)\right.\right. \\
&-\left.\left. g\left(s, x_2(h_1(s)), \int_0^s a\left(s, \tau, x_2(h_2(\tau))\right)\,\mathrm{d}\tau\right)\right)\mathrm{d}s \right\|_\alpha^2 \\
&+ 4E\left\| \int_0^t AS(t-s)\int_0^s f(s-\tau)g\left(\tau, x(h_1(\tau)), \int_0^\tau a(\tau, \delta, x(h_2(\delta)))\,\mathrm{d}\delta\right.\right. \\
&-\left.\left. g\left(\tau, x(h_1(\tau)), \int_0^\tau a(\tau, \delta, x(h_2(\delta)))\,\mathrm{d}\delta\right)\mathrm{d}\tau\right)\mathrm{d}s \right\|_\alpha^2 \\
&\le 4M_2M_5(1+M_1+M_4)\sup_{s\in J} E\|x_1(s) - x_2(s)\|_\alpha^2 + 4bE\int_0^t \frac{C_{1-\beta}^2}{(t-s)^{2(1-\beta)}}\,\mathrm{d}sM_5(1+M_4)\sup_{s\in J}\|x_1(s) - x_2(s)\|_\alpha^2 \\
&+ 4b^2M_3E\int_0^t \frac{C_{1-\beta}^2}{(t-s)^{2(1-\beta)}}\,\mathrm{d}sM_5(1+M_4)\sup_{s\in J}\|x_1(s) - x_2(s)\|_\alpha^2 \\
&\le 4M_2M_5(1+M_1+M_4)\sup_{s\in J} E\|x_1(s) - x_2(s)\|_\alpha^2 \\
&+ 4bM_5(1+M_4)(1+bM_3)\frac{b^{2\beta-1}}{2\beta-1}\sup_{s\in J} E\|x_1(s) - x_2(s)\|_\alpha^2 \\
&= L_0 \sup_{s\in J} E\|x_1(s) - x_2(s)\|_\alpha^2,
\end{aligned}$$

where $L_0 = 4M_5\left[M_2(1+M_1+M_4) + b(1+M_4)(1+bM_3)\frac{b^{2\beta-1}}{2\beta-1}\right]$. Hence, we obtain

$$E\|\Phi_1(x_1) - \Phi_1(x_2)\|_\alpha^2 \le L_0E\|x_1 - x_2\|_\alpha^2.$$

Therefore, (4) shows that Φ_1 is a contraction.

Step 2. $\Phi_2 x$ is convex for each $x \in \mathcal{B}$.

In fact, if $u_1, u_2 \in \Phi_2(x)$, then, there exists $\sigma_1, \sigma_2 \in N_{F,x(h_3(t))}$ such that

$$u_i(t) = S(t)\left[x_0 - h(x)\right] + \int_0^t S(t-s)\sigma_i(s)\,\mathrm{d}w(s) + \sum_{0<t_k<t} S(t-t_k)I_k\left(x\left(t_k^-\right)\right), \quad i = 1, 2, t \in J.$$

Let $\lambda \in [0, 1]$. Then for each $t \in J$, we have

$$\begin{aligned}
(\lambda u_1(t) + (1-\lambda)u_2(t)) &= S(t)\left[x_0 - h(x)\right] + \int_0^t S(t-s)\left[\lambda\sigma_1(s) + (1-\lambda)\sigma_1(s)\right]\mathrm{d}w(s) \\
&+ \sum_{0<t_k<t} S(t-t_k)I_k\left(x\left(t_k^-\right)\right).
\end{aligned}$$

Since $N_{F,x(h_3(t))}$ is convex (because F has convex values), then, we have $\lambda u_1(t) + (1-\lambda)u_2(t) \in \Phi_2(x)$.

Step 3. Φ_2 maps bonded sets into bounded sets in $\mathcal{B}$.

Indeed, it is enough to show that there exists a positive constant Λ such that for each $u \in \Phi_2(x)$, $x \in B_l$, we have $E\|u(t)\|_\alpha^2 \le \Lambda$.

If $u \in \Phi_2(x)$, then there exists $\sigma \in N_{F,x(h_3(t))}$ such that, for each $t \in J$

$$u(t) = S(t)\left[x_0 - h(x)\right] + \int_0^t S(t-s)\sigma(s)\,\mathrm{d}w(s) + \sum_{0<t_k<t} S(t-t_k)I_k\left(x\left(t_k^-\right)\right).$$

Therefore, by the assumptions, for each $t \in J$, we have

$$\begin{aligned} E\|u(t)\|_\alpha^2 &\le 4E\|S(t)x_0\|_\alpha^2 + 4E\|S(t)h(x)\|_\alpha^2 + 4E\left\|\int_0^t S(t-s)\sigma(s)\,\mathrm{d}w(s)\right\|_\alpha^2 + 4E\left\|\sum_{0<t_k<t} S(t-t_k)I_k\left(x\left(t_k^-\right)\right)\right\|_\alpha^2 \\ &\le 4M_1E\|x_0\|_\alpha^2 + 4M_1\left(M_6 l + M_7\right) + 4M_1\mathrm{Tr(Q)}b\mu(l) + 4M_1m^2\sum_{k=1}^m J_k(l) \\ &:= \Lambda. \end{aligned}$$

Then, for each $u \in \Phi_2(x)$, we have $E\|u(t)\|_\alpha^2 \le \Lambda$.

Step 4. Φ_2 maps bounded sets into equicontinuous sets of $\mathcal{B}$.

Let $0 < \tau_1 < \tau_2 \le b$. Then, we have for each $x \in B_l$ and $u \in \Phi_2(x)$, there exists $\sigma \in N_{F,x(h_3(t))}$ such that, for each $t \in J$, we have

$$u(t) = S(t)\left[x_0 - h(x)\right] + \int_0^t S(t-s)\sigma(s)\,\mathrm{d}w(s) + \sum_{0<t_k<t} S(t-t_k)I_k\left(x\left(t_k^-\right)\right).$$

Then, we have

$$\begin{aligned} E\|u(\tau_2) - u(\tau_1)\|_\alpha^2 \le{}& 6E\|\left[S(\tau_1) - S(\tau_2)\right](x_0 - h(x))\|_\alpha^2 + 6E\left\|\int_0^{\tau_1-\varepsilon}\left[S(\tau_2 - s) - S(\tau_1 - s)\right]\sigma(s)\,\mathrm{d}w(s)\right\|_\alpha^2 \\ &+ 6E\left\|\int_{\tau_1-\varepsilon}^{\tau_1}\left[S(\tau_2 - s) - S(\tau_1 - s)\right]\sigma(s)\,\mathrm{d}w(s)\right\|_\alpha^2 + 6E\left\|\int_{\tau_1}^{\tau_2} S(\tau_2 - s)\sigma(s)\,\mathrm{d}w(s)\right\|_\alpha^2 \\ &+ 6E\left\|\sum_{0<t_k<\tau_1}\left[S(\tau_2 - t_k) - S(\tau_1 - t_k)\right]I_k\left(x_{t_k}^-\right)\right\|_\alpha^2 + 6E\left\|\sum_{\tau_1\le t_k<\tau_2} S(\tau_2 - t_k)I_k\left(x_{t_k}^-\right)\right\|_\alpha^2 \\ \le{}& 6E\|\left[S(\tau_1) - S(\tau_2)\right](x_0 - h(x))\|_\alpha^2 + 6b\mathrm{Tr(Q)}E\int_0^{\tau_1-\varepsilon}\|A^\alpha\left[S(\tau_2 - s) - S(\tau_1 - s)\right]\|^2\,w(l)\,\mathrm{d}s \\ &+ 6b\mathrm{Tr(Q)}E\int_{\tau_1-\varepsilon}^{\tau_1}\|A^\alpha\left[S(\tau_2 - s) - S(\tau_1 - s)\right]\|^2\,\mu(l)\,\mathrm{d}s + 6b\mathrm{Tr(Q)}E\int_{\tau_1}^{\tau_2}\|S(\tau_2 - s)\sigma(s)\|^2\,\mathrm{d}s \\ &+ 6\sum_{0<t_k<\tau_1}\|\left[S(\tau_2 - t_k) - S(\tau_1 - t_k)\right]\|^2 J_k(l) + 6M_1m^2(\tau_2 - \tau_1)^2\sum_{k=1}^m J_k(l). \end{aligned}$$

The right-hand side of the above inequality tends to zero as $\tau_2 \to \tau_1$ with ε sufficiently small, since $S(t)$ is strongly continuous and the compactness of $S(t)$ for $t > 0$ implies $S(t)$, $A^\alpha S(t)$ the continuity in the uniform operator topology. Thus, the set $\{\Phi_2(x) : x \in B_l\}$ is equicontinuous.

Step 5. $(\Phi_2 B_l)(t)$ is relatively compact in H_α for each $t \in J$, where $(\Phi_2 B_l)(t) = \{u(t) : u \in \Phi_2 B_l\}$, $t \in J$.

By (H7), the set $(\Phi_2 B_l)(t)$ is relatively compact in $\mathcal{B}$ for $t = 0$. Let $0 < t \le b$ fixed and for $0 < \varepsilon < t$. For $x \in B_l$ and $u \in \Phi_2(x)$, there exists $\sigma \in N_{F,x(h_3(t))}$ such that

$$u(t) = S(t)[x_0 - h(x)] + \int_0^{t-\varepsilon} S(t-s)\sigma(s)\,\mathrm{d}w(s) + \int_{t-\varepsilon}^t S(t-s)\sigma(s)\,\mathrm{d}x(s) + \sum_{0<t_k<t} S(t-t_k)I_k\left(x\left(t_k^-\right)\right).$$

Define

$$u_\varepsilon(t) = S(t)[x_0 - h(x)] + S(\varepsilon)\int_0^{t-\varepsilon} S(t-\varepsilon-s)\sigma(s)\,\mathrm{d}w(s) + \sum_{0<t_k<t-\varepsilon} S(t-t_k)I_k\left(x\left(t_k^-\right)\right).$$

Since $S(t)$ is a compact operator, the set $V_\varepsilon(t) = \{u(t) : u \in \Phi_2(B_l)\}$ is relative compact in $\mathcal{B}$ for each ε, $0 < \varepsilon < t$. Moreover,

$$E\|u(t) - u_\varepsilon(t)\|_\alpha^2 \le 4bM_1\mu(l)\varepsilon + 4m^2M_1\sum_{t-\varepsilon<t_k<t} J_k(l).$$

Therefore, letting $\varepsilon \to 0$, we can see that there are relative compact sets arbitrarily close to the set $\{u(t) : u \in \Phi_2(B_l)\}$. Thus, the set $\{u(t) : u \in \Phi_2(B_l)\}$ is relative compact in $\mathcal{B}$. Hence, the Arzelá–Ascoli theorem shows that Φ_2 is a compact multi-valued map.

Step 6. Φ_2 has a closed graph.

Let $x_n \to x_*$, $x_n \in B_l$, $u_n \in \Phi_2(x_n)$ and $u_n \to u_*$. We aim to show that $u_* \in \Phi_2(x_*)$. Indeed, $u_n \in \Phi_2(x_n)$ means that there exists $\sigma_n \in N_{F,x_n(h_3(t))}$ such that

$$u_n(t) = S(t)[x_0 - h(x_n)] + \int_0^t S(t-s)\sigma_n(s)\,\mathrm{d}w(s) + \sum_{0<t_k<t} S(t-t_k)I_k\left(x_n\left(t_k^-\right)\right), \quad t \in J.$$

We must prove that there exists $\sigma_* \in N_{F,x_*(h_3(t))}$ such that

$$u_*(t) = S(t)[x_0 - h(x_*)] + \int_0^t S(t-s)\sigma_*(s)\,\mathrm{d}w(s) + \sum_{0<t_k<t} S(t-t_k)I_k\left(x_*\left(t_k^-\right)\right), \quad t \in J.$$

Since I_k, $k = 1, 2, \ldots, m$, and h are continuous, we have

$$\left\| \left(u_n(t) + S(t)[x_0 - h(x_n)] - \sum_{0<t_k<t} S(t-t_k)I_k\left(x_n\left(t_k^-\right)\right)\right) - \left(u_*(t) + S(t)[x_0 - h(x_*)] - \sum_{0<t_k<t} S(t-t_k)I_k\left(x_*\left(t_k^-\right)\right)\right)\right\|_{\mathcal{B}}^2 \to 0, \quad \text{as } n \to \infty.$$

Considering the linear continuous operator

$$\Gamma : L^2(J, H) \to C(J, H), \quad \sigma : \Gamma(\sigma)(t) = \int_0^t S(t-s)\sigma(s)\,\mathrm{d}w(s),$$

from Lemma 3, it follows that $\Gamma \circ N_F$ is a closed graph operator. Furthermore, we have

$$u_n(t) + S(t)[x_0 - h(x_n)] - \sum_{0<t_k<t} S(t-t_k)I_k\left(x_n\left(t_k^-\right)\right) \in \Gamma(N_{F,x_n(h_3(t))}).$$

Since $x_n \to x_*$, it follows from Lemma 3 that

$$u_*(t) + S(t)[x_0 - h(x_*)] - \sum_{0<t_k<t} S(t-t_k)I_k\left(x_*\left(t_k^-\right)\right) \in \Gamma\left(N_{F,x_*(h_3(t))}\right).$$

That is, there exists a $\sigma_* \in \Gamma\left(N_{F,x_*(h_3(t))}\right)$ such that

$$u_*(t) + S(t)[x_0 - h(x_*)] - \sum_{0<t_k<t} S(t-t_k)I_k\left(x_*\left(t_k^-\right)\right) = \Gamma\left(\sigma_*(t)\right) = \int_0^t S(t-s)\sigma_*(s)\,\mathrm{d}w(s).$$

Therefore, Φ_2 has a closed graph and therefore Φ_2 is u.s.c.

Step 7. The operator inclusion $x \in \Phi_1(x) + \Phi_2(x)$ has a solution in $B[0, r]$.

Define an open ball $B(0, r)$ in $\mathcal{B}$, where r satisfies the inequality given in (H8). As a consequence of the above steps, we know that Φ_1 and Φ_2 satisfy all conditions of Theorem 4. Therefore, if we can show that the second assertion of Theorem 4 is not true, then, we show that the system (1) has at least one mild solution. Let $u \in \mathcal{B}$ be a possible solution for $\lambda u \in \Phi_1 u + \Phi_2 u$ for some $\lambda > 1$ with $E\|u\|_\alpha^2 = r$. Then, we have

$$\begin{aligned} u(t) = {} & \lambda^{-1} g\left(t, x(h_1(t)), \int_0^t a(t, s, x(h_2(s)))\,\mathrm{d}s\right) + \lambda^{-1} S(t)\left[x_0 - h(x) - g(0, x(h_1(0)), 0)\right] \\ & + \lambda^{-1} \int_0^t AS(t-s) g\left(s, x(h_1(s)), \int_0^s a(s, \tau, x(h_2(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s \\ & + \lambda^{-1} \int_0^t AS(t-s) \int_0^s f(s-\tau) g\left(\tau, x(h_1(\tau)), \int_0^\tau a(\tau, \delta, x(h_2(\delta)))\,\mathrm{d}\delta\right)\mathrm{d}\tau\,\mathrm{d}s \\ & + \lambda^{-1} \int_0^t S(t-s)\sigma(s)\,\mathrm{d}w(s) + \lambda^{-1} \sum_{0<t_k<t} S(t-t_k)I_k\left(x\left(t_k^-\right)\right). \end{aligned}$$

Then, by the assumptions, we get

$$E\|u(t)\|_\alpha^2 \le 18\left[E\|x_0\|_\alpha^2 + (M_6 E\|u\|_\alpha^2 + M_7) + 2M_2(M_5 E\|u\|_\alpha^2 + c_2)\right]$$

$$
\begin{aligned}
&+12M_2\left[M_5(E\|u\|_\alpha^2+2M_4E\|u\|_\alpha^2+2c_1)+c_2\right]\\
&+12b\int_0^t \frac{C_{1-\beta}^2}{(t-s)^{2(1-\beta)}}\left[M_5(E\|u\|_\alpha^2+2M_4E\|u\|_\alpha^2+2c_1)+c_2\right]\mathrm{d}s\\
&+12M_3b^2\int_0^t \frac{C_{1-\beta}^2}{(t-s)^{2(1-\beta)}}\left[M_5(E\|u\|_\alpha^2+2M_4E\|u\|_\alpha^2+2c_1)+c_2\right]\mathrm{d}s\\
&+6b\mathrm{Tr(Q)}\int_0^t \frac{C_\alpha^2}{(t-s)^{2\alpha}}\mu\left(E\|u\|_\alpha^2\right)\,\mathrm{d}s+6m^2M_1\sum_{k=1}^m J_k\left(E\|u\|_\alpha^2\right).
\end{aligned}
$$

Taking the supremum over t, we obtain

$$
\begin{aligned}
E\|u\|_\alpha^2 \le\ & 18\left[E\|x_0\|_\alpha^2+(M_6E\|u\|_\alpha^2+M_7)+2M_2(M_5E\|u\|_\alpha^2+c_2)\right]\\
&+12M_2\left[M_5(E\|u\|_\alpha^2+2M_4E\|u\|_\alpha^2+2c_1)+c_2\right]\\
&+12b\int_0^t \frac{C_{1-\beta}^2}{(t-s)^{2(1-\beta)}}\left[M_5(E\|u\|_\alpha^2+2M_4E\|u\|_\alpha^2+2c_1)+c_2\right]\mathrm{d}s\\
&+12M_3b^2\int_0^t \frac{C_{1-\beta}^2}{(t-s)^{2(1-\beta)}}\left[M_5(E\|u\|_\alpha^2+2M_4E\|u\|_\alpha^2+2c_1)+c_2\right]\mathrm{d}s\\
&+6b\mathrm{Tr(Q)}\int_0^t \frac{C_\alpha^2}{(t-s)^{2\alpha}}\mu\left(E\|u\|_\alpha^2\right)\,\mathrm{d}s+6m^2M_1\sum_{k=1}^m J_k\left(E\|u\|_\alpha^2\right).
\end{aligned}
$$

Substituting $E\|u\|_\alpha^2=r$ in the above inequality and noting that (H8) holds, we get

$$
r\le \frac{L_1+\frac{6b^{1-2\alpha}}{1-2\alpha}\mathrm{Tr(Q)}\mu(r)+6m^2M_1\sum\limits_{k=1}^m J_k(r)}{1-L_2},
$$

which is contradiction to (H8). Thus, the operator inclusions $x\in\Phi_1(x)+\Phi_2(x)$ has a solution in $B[0,r]$. Therefore, the system (1) has at least one mild solution x in $\mathcal{B}$. □

4. An example

Consider the nonlinear impulsive neutral stochastic functional integro-differential inclusion of the following form:

$$
\begin{cases}
\dfrac{\partial}{\partial t}\left[v(t,x)-\displaystyle\int_0^\pi a(t,x,\theta)v(\sin t,\theta)\,\mathrm{d}\theta\right]-\dfrac{\partial^2}{\partial x^2}\left[v(t,x)+\displaystyle\int_0^t b(t-s)v(s,x)\,\mathrm{d}s\right]\\
\quad\in\kappa\left(t,\dfrac{\partial}{\partial x}v(\sin t,x)\right)\mathrm{d}w(t),\quad 0\le t\le 1,\ t\ne t_k,\\
\Delta v(t_k,x)=v(t_k^+,x)-v(t_k^-,x)=I_k\left(v\left(t_k^-,x\right)\right),\quad k=1,2,\dots,m,\\
v(t,0)=v(t,\pi)=0,\quad 0\le t\le 1,\\
v(0,x)+\displaystyle\int_0^\pi k(x,\theta)\,\mathrm{d}\theta=u_0(x),\quad 0\le x\le\pi,
\end{cases}
\tag{6}
$$

where $w(t)$ denotes a one-dimensional standard Wiener process, $a:[0,1]\times[0,\pi]\times[0,\pi]\to\mathbf{R}$, $\kappa:[0,1]\times\mathbf{R}\to\mathcal{P}(\mathbf{R})$, $k:[0,\pi]\times[0,\pi]\to\mathbf{R}$, $b(t)$, $t\in\mathbf{R}$ are continuous functions and there exists a constant K_1 such that $|b(t-s)|\le K_1$ and $u_0(\cdot)\in L^2([0,\pi])$ is $\mathcal{F}_0$-measurable and satisfies $E\|u_0\|^2<\infty$.

Let $H=L^2([0,\pi])$ with the norm $\|\cdot\|$. Define $A:H\to H$ by $Az=z''$ with domain

$$
D(A)=\left\{z\in H, z, z' \text{ are absolutely continuous } z''\in H, z(0)=z(\pi)=0\right\}.
$$

Then,

$$
Az=\sum_{n=1}^\infty n^2(z,z_n),\quad z\in D(A),
$$

where $z_n(s)=\sqrt{\frac{2}{\pi}}\sin(ns)$, $n=1,2,\dots$ is the orthogonal set of eigenvectors in A. It is well known that A generates a strongly continuous semigroup that is analytic, and resolvent operator $S(t)$ can be extracted this analytic semigroup and give by

$$
S(t)z=\sum_{n=1}^\infty \exp^{-n^2t}(z,z_n)z_n,\quad z\in H.
$$

Since the analytic semigroup $S(t)$ is compact, there exists a constant M_1 such that $\|S(t)\|^2 \leq M_1$. Especially, the operator $A^{1/2}$ is given by

$$A^{1/2}\omega = \sum_{n=1}^{\infty} n \langle \omega, z_n \rangle z_n$$

with the domain $D\left(A^{1/2}\right) = \{\omega \in H : \sum_{n=1}^{\infty} n \langle \omega, z_n \rangle z_n\}$.

Hence, let

$$g(t, v)(\cdot) = \int_0^{\pi} a(t, \cdot, \theta) v(\theta)\, \mathrm{d}\theta, \qquad F(t, v)(\cdot) = \kappa(t, v'(\cdot))$$

and

$$h(\omega)(\cdot) = \int_0^{\pi} k(\cdot, \theta)\omega(\theta)\, \mathrm{d}\theta, \quad \omega \in \mathcal{B}.$$

Let $h_1(t) = h_3(t) = \sin t$. Then, the system (6) takes the following abstract form:

$$\begin{cases} \mathrm{d}\left[x(t) - g(t, x(h_1(t)))\right] \in A\left[x(t) + \int_0^t f(t-s)x(s)\,\mathrm{d}s\right] \mathrm{d}t \\ \quad + F(t, x(h_3(t)))\,\mathrm{d}w(t), \quad t \in J := [0, b],\, t \neq t_k, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k\left(x\left(t_k^-\right)\right), \quad k = 1, 2, \ldots, m, \\ x(0) + h(x) = x_0. \end{cases} \tag{7}$$

Moreover, $g : J \times H_{1/2} \to H_{1/2},\ A^{1/2}g : J \times H_{1/2} \to H_{1/2}$ and $F : J \times H_{1/2} \to L(K, H)$. Further, we can impose some suitable conditions on the above-defined functions to verify the assumptions on Theorem 7, we can conclude that system (6) has at least one mild solution on J.

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